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The large N limit of exceptional Jordan matrix models and M, F theory

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Abstract

The large $N \to \infty$ limits of the exceptional F_4, E_6 Jordan matrix models of Smolin and Ohwashi lead to *novel* Chern–Simons membrane Lagrangians which are suitable candidates for providing a nonperturbative *bosonic* formulation of M theory in $D = 27$ real and complex dimensions, respectively. Freudenthal algebras and *triple* Freudenthal products permit the construction of a *novel* $E_7 \times SU(N)$ invariant matrix model whose large N limit yields generalized nonlinear sigma model actions on 28-complex-dimensional backgrounds associated with a 56-real-dimensional phase space realization of the Freudenthal algebra. We argue as to why the latter matrix model, in the large N limit, might be the proper arena for a bosonic formulation of F theory. Finally, we display generalized Dirac–Nambu–Goto membrane actions in terms of $3 \times 3 \times 3$ cubic matrix entries that match the numbers of degrees of freedom of the 27-dimensional exceptional Jordan algebra $J_3[0]$.

Keywords: Jordan; Division algebras; Branes; Matrix models; M, F theory

1. The large N limit of exceptional Jordan matrix models and 27-dimensional bosonic M theory

Not long ago it was shown that p-brane actions (when $p + 1 = 4k$) could be obtained from the large N limit of $SU(N)$ Yang–Mills and generalized Yang–Mills theories in flat backgrounds [4]. A Moyal deformation quantization was instrumental in the construction of p-brane actions and Chern–Simons branes from the large N limit of $SU(N)$ Yang–Mills in flat backgrounds. This Moyal deformation approach also furnishes dynamical membranes (a QCD membrane) as well when one uses the spatial quenching approximation to a *line* (one dimension), instead of quenching to a point. The large N limit of non-Abelian $SU(N)$ Born–Infeld models and its relation to Nambu–Goto–Dirac string actions was also obtained in [5].

In order to take the large N limit of exceptional Jordan matrix models, which yields a *novel* Chern–Simons membrane Lagrangian as a suitable candidate Lagrangian for providing a nonperturbative formulation of the *bosonic* sector of M theory in $D = 27$ real, complex dimensions, respectively [10], one must start by explaining briefly how a $4D$ $SU(N)$ Yang–Mills theory, reduced and quenched to a point, leads to p-branes and Chern–Simons brane actions

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in the large N limit [4]. In [2] the large N limit of $SU(N)$ was identified with the area-preserving diffeomorphisms algebra of a sphere. The $SU(N)$ Yang–Mills action reduced and quenched to a point is

$$S = -\frac{1}{4}\left(\frac{2\pi}{a}\right)^4 \frac{N}{g_{YM}^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}).$$
$$F_{\mu\nu} = [iD_\mu, iD_\nu] \tag{1.1}$$

where a is the lattice-cell scale. Notice that the reduced–quenched action is defined at a "point" x_o. The quenched approximation is based essentially on replacing the field strengths by their commutator, dropping the ordinary derivative terms. For simplicity we have omitted the matrix $SU(N)$ indices in (1–1). The *theta* term in QCD is

$$S_\theta = -\frac{\theta N g_{YM}^2}{16\pi^2}\left(\frac{2\pi}{a}\right)^4 \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}). \tag{1.2}$$

The WWGM (Weyl–Wigner–Groenowold–Moyal) quantization establishes a one-to-one correspondence between a linear operator $D_\mu = \partial_\mu + A_\mu$ acting on the Hilbert space $\mathcal{H}$ of square integrable functions in R^D and a smooth function $\mathcal{A}_\mu(x, y)$ which is the Fourier transform of $\mathcal{A}_\mu(q, p)$. The latter quantity is obtained by evaluating the trace of the $D_\mu = \partial_\mu + A_\mu$ operator summing over the diagonal elements with respect to an orthonormal basis in the Hilbert space. Under the WWGM correspondence, in the quenched–reduced approximation, the matrix product $A_\mu \cdot A_\nu$ is mapped into the *noncommutative* Moyal star product of their symbols $\mathcal{A}_\mu * \mathcal{A}_\nu$ and the commutators are mapped into their Moyal brackets:

$$\frac{1}{i\hbar}[A_\mu, A_\nu] \Rightarrow \frac{1}{i\hbar}\{\mathcal{A}_\mu, \mathcal{A}_\nu\}_{MB} \to \{\mathcal{A}_\mu, \mathcal{A}_\nu\}_{PB} \quad \text{when } \hbar \to 0. \tag{1.3}$$

Replacing the trace operation with an integration w.r.t. the internal phase space variables, $\sigma \equiv q^i, p^i$, gives

$$\frac{(2\pi)^4}{N^4}\mathrm{Trace} \to \int \mathrm{d}^4\sigma. \tag{1.4}$$

The WWGM deformation quantization of the quenched–reduced original actions is

$$S^* = -\frac{1}{4}\left(\frac{2\pi}{a}\right)^4 \frac{N}{g_{YM}^2}\int \mathrm{d}^4\sigma \mathcal{F}_{\mu\nu}(\sigma) * \mathcal{F}^{\mu\nu}(\sigma).$$
$$\mathcal{F}_{\mu\nu} = \{i\mathcal{A}_\mu, i\mathcal{A}_\nu\} \tag{1.5}$$

and the corresponding WWGM deformation of the *theta* term is

$$S_\theta^* = -\frac{\theta N g_{YM}^2}{16\pi^2}\left(\frac{2\pi}{a}\right)^4 \epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}^4\sigma \mathcal{F}_{\mu\nu}(\sigma) * \mathcal{F}_{\rho\sigma}(\sigma). \tag{1.6}$$

By taking the following gauge field/coordinate correspondence:

$$\mathcal{A}_\mu(\sigma) \to \left(\frac{2\pi}{N}\right)^{1/4} X_\mu(\sigma)$$
$$\mathcal{F}_{\mu\nu}(\sigma) \to \left(\frac{2\pi}{N}\right)^{1/2} \{X_\mu(\sigma), X_\nu(\sigma)\} \tag{1.7}$$

and, finally, by setting the Moyal deformation parameter "$\hbar$" $= 2\pi/N$ of the WWGM deformed action to zero, i.e. by taking the classical $\hbar = 0$ limit, which is tantamount to taking the $N = \infty$ limit, one can see that the quenched–reduced YM action in the large N limit will become the Dolan–Tchrakian action for a 3-brane, in the *conformal gauge* [3], moving in a *flat* $D = 4$-dimensional (hereafter, 'dim') background [4]:

$$S = -\frac{1}{4g_{YM}^2}\left(\frac{2\pi}{a}\right)^4 \int \mathrm{d}^4\sigma \{X^\mu, X^\nu\}_{PB}\{X^\rho, X^\tau\}_{PB}\eta_{\mu\rho}\eta_{\nu\tau} + \text{ permutations} \tag{1.8}$$

due to the fact that the Moyal brackets times $\frac{1}{\hbar}$ collapse to the ordinary Poisson brackets in the $\hbar = 2\pi/N = 0$ limit (large N limit), whereas the action corresponding to the *theta* term will become in the $N = \infty$ limit the Chern–Simons Zaikov action for a closed membrane embedded in a four-dim (pseudo-)Euclidean background and whose three-dim world-volume is the boundary of the four-dim hadronic bag. The Chern–Simons membrane has *nontrivial* boundary dynamics compared with the trivial bulk dynamics of the spacetime filling 3-brane.

This introductory review is essential for understanding how to related the large N limit of the exceptional matrix models [7,6] to a novel version of Chern–Simons branes. Ohwashi [6] defined his E_6 matrix model by starting with the matrix $\mathcal{M}^A$ elements of the Jordan $J_3[O]$ algebra $\mathcal{J} \times \mathcal{G}$ [12,17,16]:

$$\begin{pmatrix} \mathcal{A}_1^A & \Phi_3^A & \bar{\Phi}_2^A \\ \bar{\Phi}_3^A & \mathcal{A}_2^A & \Phi_1^A \\ \Phi_2^A & \bar{\Phi}_1^A & \mathcal{A}_3^A \end{pmatrix}$$

where $\mathcal{J}$ is the complexified Jordan algebra of degree 3, $J_3[C \times O]$, and $\mathcal{G}$ is the $u(N)$ Lie algebra corresponding to the $U(N)$ group with structure constants f_{ABC}. $[T_A, T_B] = f_{ABC} T_C$. $\mathcal{A}_I (I = 1, 2, 3)$ are complex-valued numbers and Φ_I are elements of the *complex* Graves–Cayley octonion algebra comprised of complex octonions $(x_o + \mathbf{i} y_o) e_o + (x_i + \mathbf{i} y_i) e_i$; $i = 1, 2, 3, \ldots, 7$. The bar operation $\bar{\Phi}$ denotes the octonionic conjugation $(x_o + \mathbf{i} y_o) e_o - (x_i + \mathbf{i} y_i) e_i$ that must not be confused with complex conjugation, $(x_o - \mathbf{i} y_o) e_o + (x_i - \mathbf{i} y_i) e_i$. The action of Ohwashi was based on the cubic form

$$S = (\rho^2(\mathcal{M}^{[A}), \rho(\mathcal{M}^B), \mathcal{M}^{C]}) f_{ABC} \qquad (X, Y, Z) = \mathrm{tr}(X \cdot (Y \times_F Z)) \tag{1.9}$$

where $\rho, \rho^3 = 1$ is the cycle mapping (based on the *triality* symmetry of $SO(8)$) that takes the index $I \to I + 1$, modulo 3. It is essential to introduce the cycle mapping in (1.9); otherwise the expression would have been identically equal to zero due to the fact that the cubic form is symmetric in its three entries while f_{ABC} is antisymmetric. The product $Y \times_F Z$ is the *symmetric* Freudenthal product

$$Y \times_F Z = Y \cdot Z - \frac{1}{2}\mathrm{tr}(Y) Z - \frac{1}{2}\mathrm{tr}(Z) Y + \frac{1}{2}\mathrm{tr}(Y)\mathrm{tr}(Z) - \frac{1}{2}\mathrm{tr}(Y \cdot Z)\mathbf{1} \tag{1.10}$$

and $X \cdot Y$ is the commutative but non-associative Jordan product given by the anticommutator $\frac{1}{2}(XY + YX)$ obeying the Jordan identity $(X \cdot Y) \cdot X^2 = X \cdot (Y \cdot X^2)$. The cubic form (1.9) is very different from the trilinear form $\mathrm{tr}(X \cdot (Y \cdot Z))$ used by Smolin [7] to construct the F_4 matrix model based on $J_3[O]$ rather than $J_3[C \times O]$. The action of Ohwashi is complex valued while that of Smolin is real valued. The explicit evaluation of the expression (1.9) can be found in [6] where he includes a detailed appendix with numerous important formulae that are indispensable for writing down all the explicit terms of the cubic form. The action is

$$\begin{aligned} S = \mathrm{Trace}\Bigg[& \epsilon^{IJK} (\mathbf{A}_I[\mathbf{A}_J, \mathbf{A}_K] + \eta^{ij} \boldsymbol{\Phi}_{0I}[\boldsymbol{\Phi}_{iJ}, \boldsymbol{\Phi}_{jK}] + \sigma^{ijk} \boldsymbol{\Phi}_{iI}[\boldsymbol{\Phi}_{jJ}, \boldsymbol{\Phi}_{kK}] \\ & + \eta^{ij} \mathbf{A}_I[\boldsymbol{\Phi}_{iJ}, \boldsymbol{\Phi}_{jK}] + \mathbf{A}_I[\boldsymbol{\Phi}_{0J}, \boldsymbol{\Phi}_{0K}] + \boldsymbol{\Phi}_{0I}[\boldsymbol{\Phi}_{0J}, \boldsymbol{\Phi}_{0K}]) + \sigma^{ijk} \sum_{I=1}^{I=3} \boldsymbol{\Phi}_{iI}[\boldsymbol{\Phi}_{jI}, \boldsymbol{\Phi}_{kI}] \Bigg] \end{aligned} \tag{1.11}$$

where the matrix-valued quantities are denoted by the bold-face letters: $\mathbf{A}_I = \mathcal{A}_I^A T_A$; $\boldsymbol{\Phi}_{iJ} = \Phi_{iJ}^A T_A$, etc.

We will describe now how to take the large N limit of the E_6 exceptional matrix model action described by Eq. (1.11) [6]. To achieve this goal one needs to follow steps similar to those taken in Eqs. (1.1)–(1.8) to relate the large N limit of quenched–reduced $SU(N)$ Yang–Mills actions to strings, membranes and 3-brane (bag) actions [4]. Upon doing so one arrives at the following correspondence among each single one of the terms of the matrix model action Eq. (1.11) and the following terms of the Chern–Simons membrane model that are displayed next, for example, the large N limit of

$$\lim_{N\to\infty} \epsilon^{IJK} \mathrm{tr}_{N\times N}(\mathcal{A}_I^A T_A [\mathcal{A}_J^B T_B, \mathcal{A}_K^C T_C]) \to \int [\mathrm{d}^2 \Sigma]_a \epsilon^{abc} \mathcal{A}_I (\partial_b \mathcal{A}_J)(\partial_c \mathcal{A}_K) \epsilon^{IJK} \tag{1.12}$$

where the N^2 matrices $T_A, T_B, T_C \ldots$ are $N \times N$ Hermitian matrices associated with the Lie algebra $u(N)$ generators corresponding to the group $U(N) = SU(N) \times U(1)$. The $U(1)$ piece corresponds to the center of mass mode since the variables in the Chern–Simons brane actions must be understood in terms of $X - X(0)$, i.e. relative to an origin, in order to preserve translational invariance. The indices I, J, K run over $1, 2, 3$. The indices $a, b.c$ run over $1, 2, 3$, the three degrees of freedom of the world-volume of a membrane. The surface boundary element of the three-dim world-volume V is $\mathrm{d}^2\vec{S} = \mathrm{d}^2\Sigma_a n^a$ where n^a is a unit vector pointing in the outwards normal direction.

Following the same large N limit procedure (see [4,5]) of Eq. (1.12) that was summarized in Eqs. (1.1)–(1.8) in each single one of the remaining terms of the E_6 exceptional matrix model action (1.11) leads to the one-to-one correspondence

$$\lim_{N\to\infty} \epsilon^{IJK} \mathrm{tr}_{N\times N}(\eta^{ij}\Phi^A_{iI}T_A[\mathcal{A}^B_J T_B, \Phi^C_{jK}T_C]) \to \int [\mathrm{d}^2\Sigma]_a \epsilon^{abc}\eta^{ij}\Phi_{iI}(\partial_b\mathcal{A}_J)(\partial_c\Phi_{jK})\epsilon^{IJK} \tag{1.13a}$$

$$\lim_{N\to\infty} \epsilon^{IJK} \mathrm{tr}_{N\times N}(\sigma^{ijk}\Phi^A_{iI}T_A[\Phi^B_{jJ} T_B, \Phi^C_{kK}T_C]) \to \int [\mathrm{d}^2\Sigma]_a \epsilon^{abc}\sigma^{ijk}\Phi_{iI}(\partial_b\Phi_{jJ})(\partial_c\Phi_{kK})\epsilon^{IJK} \tag{1.13b}$$

and so forth.

Therefore, upon taking the large N limit correspondence among all of the terms of Eq. (1.11) as shown in Eqs. (1.12) and (1.13), we can show that the large N limit of the E_6 exceptional Jordan matrix model [6] is given by a Chern–Simons brane action explicitly presented below in Eq. (1.14). The crux of this large $N \to \infty$ limit correspondence relies on the fact that the $N \times N$ matrices $\mathbf{A} \to \mathcal{A}(\sigma^1, \sigma^2, \sigma^3)$ become the membrane coordinates in the continuum limit: the $\mathrm{trace}_{N\times N} \to \int$; commutators $\to$ brackets and the Jordan algebra non-associator $[X, Y, Z] = X\cdot(Y\cdot Z) - (X\cdot Y)\cdot Z$ has a correspondence with the Nambu–Poisson brackets $\{X(\sigma^a), Y(\sigma^a), Z(\sigma^a)\}$ as discussed by [17]. Similar results can be obtained in the large N limit of the F_4 matrix models of [7], with the only difference that one must use the trilinear form based on Jordan products instead of the cubic form (based on the Jordan and Freudenthal product). We urge the reader to study the procedure in [5,6] which explain rigorously why the large N limit of discrete matrix models leads to strings, membranes and p-brane actions. This procedure is what permits us to write the correspondence given by Eqs. (1.11)–(1.13).

Finally, the *bulk* three-dim action whose two-dim *boundary* action is given by the large N limit of Eq. (1.11), omitting numerical factors, is explicitly given by

$$\begin{aligned} S = \int_V \mathrm{d}^3V\epsilon^{abc}\Bigg[&\epsilon^{IJK}(\partial_a\mathcal{A}_I\partial_b\mathcal{A}_J\partial_c\mathcal{A}_K + \eta^{ij}\partial_a\Phi_{0I}\partial_b\Phi_{iJ}\partial_c\Phi_{jK} + \sigma^{ijk}\partial_a\Phi_{iI}\partial_b\Phi_{jJ}\partial_c\Phi_{kK} \\ &+ \eta^{ij}\partial_a\mathcal{A}_I\partial_b\Phi_{iJ}\partial_c\Phi_{jK} + \partial_a\mathcal{A}_I\partial_b\Phi_{0J}\partial_c\Phi_{0K} + \partial_a\Phi_{0I}\partial_b\Phi_{0J}\partial_c\Phi_{0K}) + \sigma^{ijk}\sum_{I=1}^{I=3}\partial_a\Phi_{iI}\partial_b\Phi_{jI}\partial_c\Phi_{kI}\Bigg]. \end{aligned} \tag{1.14}$$

Concluding, the $N \to \infty$ limit of the Ohwashi E_6 matrix model action (1.11) is given by a *novel* Chern–Simons membrane model (not to be confused with the Zaikov Chern–Simons membrane [1]).

The action can be written in condensed form as

$$S = \int_V [\mathrm{d}^3V]\epsilon^{abc}(\partial_a\mathbf{J}, \partial_b\rho(\mathbf{J}), \partial_c\rho^2(\mathbf{J})). \tag{1.15}$$

The above action can also be recast in terms of Nambu–Poisson brackets as

$$\begin{aligned} \int \mathrm{d}^3\sigma\Bigg[&\epsilon^{IJK}(\{\mathcal{A}_I, \mathcal{A}_J, \mathcal{A}_K\} + \eta^{ij}\{\Phi_{0I}, \Phi_{iJ}, \Phi_{jK}\} + \sigma^{ijk}\{\Phi_{iI}, \Phi_{jJ}, \Phi_{kK}\} \\ &+ \eta^{ij}\{\mathcal{A}_I, \Phi_{iJ}, \Phi_{jK}\} + \{\mathcal{A}_I, \Phi_{0J}, \Phi_{0K}\} + \{\Phi_{0I}, \Phi_{0J}, \Phi_{0K}\}) + \sigma^{ijk}\sum_{I=1}^{I=3}\{\Phi_{iI}, \Phi_{jI}, \Phi_{kI}\}\Bigg]. \end{aligned} \tag{1.16}$$

The integrand is a total derivative that can be integrated over a two-dim boundary domain $\Sigma \equiv \partial V$ giving

$$S = \int_{\partial V}[\mathrm{d}^2\Sigma]_a\epsilon^{abc}\Bigg[\epsilon^{IJK}(\mathcal{A}_I\partial_b\mathcal{A}_J\partial_c\mathcal{A}_K + \eta^{ij}\Phi_{0I}\partial_b\Phi_{iJ}\partial_c\Phi_{jK} + \sigma^{ijk}\Phi_{iI}\partial_b\Phi_{jJ}\partial_c\Phi_{kK}$$

$$+ \ \eta^{ij}\mathcal{A}_I \partial_b \Phi_{iJ} \partial_c \Phi_{jK} + \mathcal{A}_I \partial_b \Phi_{0J} \partial_c \Phi_{0K} + \Phi_{0I} \partial_b \Phi_{0J} \partial_c \Phi_{0K}) + \sigma^{ijk} \sum_{I=1}^{I=3} \Phi_{iI} \partial_b \Phi_{jI} \partial_c \Phi_{kI} \Bigg]. \tag{1.17}$$

The novel Chern–Simons action (1.17) (which is the large N limit of the action in Eq. (1.11)) is associated with the two-dim boundary of an open three-dim region, the world-volume of an open membrane, and is a candidate action for a nonperturbative *bosonic* formulation of M theory in $D = 27$ dimensions [10,13]. We can recognize that the action (1.17) is roughly of the same **BF** form as the topological open membrane actions in a constant antisymmetric tensor $B_{\mu_1\mu_2\mu_3}$ background [14].

In the Smolin version [7] of the action, one may interpret the 27 functions $\mathcal{A}_I(\sigma^a)$, $\Phi_{0I}(\sigma^a)$, $\Phi_{iI}(\sigma)$ associated with the Chern–Simons action (1.16) and (1.17) as *foliations* (maps) from the three-dim world-volume of the membrane (and its boundary) to the 27-dim target spacetime M^{27}. Upon foliating the three-dim world-volume V into $\Sigma \times R$, where the leaves Σ are two-dim Riemann surfaces, and R represents the world-volume clock, yields the embedding maps $X : \Sigma \times R \to \Upsilon^{16} \times M^{11}$, where the "horizontal" leaves Υ^{16} foliate the underlying space M^{27} along the "vertical" directions M^{11}. Given a *finite* size three-dim tubular region V with two boundary components $\partial V = \Sigma_1 \cup \Sigma_2$, one may interpret the first and last leaves of the foliations of M^{27} as two spatial 16-dim regions Υ^{16} such that $X : V_{\text{bulk}} \to M^{27}$ and $X : \partial V = \Sigma_1 \cup \Sigma_2 \to \Upsilon_1^{16} \cup \Upsilon_2^{16}$. This foliation picture has been studied by Zois [8], in a very different context, in order to describe the dynamics of a continuum of $N = \infty$ parallel D-branes as leaves of foliations of the underlying bulk space time with the purpose of understanding the noncommutative topology of M theory.

When the coordinates X belong to the exceptional Jordan $J_3[O]$ algebra one must use the Smolin version of the action [7] instead of the *complexified* Ohwashi version [6] and the boundary maps are represented by

$$X(\partial V) : \partial V = \Sigma_1 \cup \Sigma_2 \to (R \times O)P^2 \cup (R \times O)P^2 \tag{1.18}$$

where the projective Moufang plane is the coset space $(R \times O)P^2 \equiv F_4/\text{Spin}(9)$. Since ∂V is made out of *compact* regions (Riemann surfaces) Σ_1, Σ_2, the continuous maps $X(\partial V)$ should map compact sets into compact sets which is compatible by viewing the projective planes as compact domains of the spatial Υ^{16} regions (noncompact hypersurfaces in general). Because the isometry group of $(R \times O)P^2$ is also the automorphism group of the Jordan algebra $J_3[O]$ given by F_4, this picture of foliations is indeed consistent with the chosen boundaries since F_4 acts naturally on the projective plane $(R \times O)P^2$ via isometries.

The split of the $27 = 3 + 8 + 16$ coordinates corresponding to the entries of an element of the $J_3[O]$ algebra can be understood as follows: The three functions $\mathcal{A}_1(\sigma^a)$, $\mathcal{A}_2(\sigma^a)$, $\mathcal{A}_3(\sigma^a)$ are associated with the three longitudinal directions corresponding to the $2 + 1 = 3$-dim world-volume of the membrane. The eight functions Φ_{01}, Φ_{i1} $(i = 1, 2, 3, \dots, 7)$ represent the eight transverse directions of the membrane with respect to an 11-dim domain region inside the 27-dim bulk of M^{27}. The 16 (real) functions Φ_{02}, Φ_{i2}, Φ_{03}, Φ_{i3} correspond to the coordinates of the 16-real-dimensional $(R \times O)P^2$ projective plane.

The action written in the form (1.16) is clearly invariant under volume-preserving reparametrizations of the three-dim world-volume that leave invariant the Nambu–Poisson brackets. One may view the membrane as an incompressible fluid that can change its shape while maintaining its volume. The true dynamics of (1.16) reside in the two-dim *boundary* captured by the two-dim boundary action $S(\Sigma)$ in Eq. (1.17). There is also an invariance of the action under the *global* rigid E_6 transformations (a simply connected compact group) and which are encoded as automorphisms of the $J_3[C \times O]$ algebra under the transformations $\mathbf{J} \to \alpha\mathbf{J}$, where α is a 3×3 matrix whose entries are numerical constants and which leave invariant the cubic form in Eq. (1.14) as follows:

$$\begin{aligned} S(\boldsymbol{\alpha}\mathbf{J}) &= \int_V [\mathrm{d}^3 V] \epsilon^{abc} (\partial_a(\boldsymbol{\alpha} J), \partial_b(\alpha\rho(\mathbf{J})), \partial_c(\alpha\rho^2(\mathbf{J}))) \\ &= S(\mathbf{J}) = \int_V [\mathrm{d}^3 V] \epsilon^{abc} (\partial_a \mathbf{J}, \partial_b \rho(\mathbf{J}), \partial_c \rho^2(\mathbf{J})) \end{aligned} \tag{1.19}$$

these E_6 global (rigid) mappings $\boldsymbol{\alpha}$ also leave invariant the Hermitian product

$$\langle \boldsymbol{\alpha} X, \boldsymbol{\alpha} Y \rangle = \langle \mathbf{X}, \mathbf{Y} \rangle = (\mathbf{X}^*, \mathbf{Y}) = \mathrm{tr}(\mathbf{X}^* \cdot \mathbf{Y}). \tag{1.20}$$

There is also symmetry under the cycle mapping ρ by construction.

To end this section we analyze what the physical meaning of the *complex*-valued action obtained by [6] is and the physical implications behind the complexifications of the coordinates corresponding to the bosonic 27-dim version of M theory. Exceptional Jordan matrix models based on the compact E_6 involve a *double* number of the required physical degrees of freedom inherent in the complex-valued action [6]. This led Ohwashi to construct an interacting pair of mirror universes within the compact E_6 matrix model and equipped with a $Sp(4, \mathbf{H})/Z_2$ symmetry based on the quaternionic-valued symplectic group. The interacting picture resembles that of the bi-Chern–Simons gravity models [21]. The complex counterpart of the Chern–Simon–Witten theory has been studied by [22] where the complex (holomorphic) analogue of the Gauss linking number for complex curves embedded in a Calabi–Yau threefold was defined.

A complexification of ordinary gravity (not to be confused with Hermitian–Kahler geometry) has been known of for a long time. Complex gravity requires that $g_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ so that now one has $g_{\nu\mu} = (g_{\mu\nu})^*$, which implies that the diagonal components of the metric $g_{z_1 z_1} = g_{z_2 z_2} = g_{\tilde{z}_1 \tilde{z}_1} = g_{\tilde{z}_2 \tilde{z}_2}$ must be real. A treatment of a non-Riemannan geometry based on a complex tangent space and involving symmetric $g_{(\mu\nu)}$ plus antisymmetric $g_{[\mu\nu]}$ metric components was first proposed by Einstein and Strauss [23] (and later on by [25]) in their unified theory of electromagnetism with gravity by identifying the EM field strength $F_{\mu\nu}$ with the antisymmetric metric $g_{[\mu\nu]}$ component.

Borchsenius [24] formulated the quaternionic extension of Einstein–Strauss unified theory of gravitation with EM by incorporating appropriately the $SU(2)$ Yang–Mills field strength into the degrees of a freedom of a quaternionic-valued metric. Oliveira and Marques [26] later on provided the octonionic gravitational extension of Borchsenius theory involving two interacting $SU(2)$ Yang–Mills fields and where the exceptional group G_2 was realized naturally as the automorphism group of the octonions.

It was shown in [27] how one could generalize octonionic gravitation into an extended relativity theory in Clifford spaces, involving poly-vector-valued (Clifford-algebra-valued) coordinates and fields, where in addition to the speed of light there is also an invariant length scale (set equal to the Planck scale) in the definition of a generalized metric distance in Clifford spaces encoding, lengths, areas, volumes and hyper-volume metrics.

Generalized complex geometry was developed by Hitchin and involves a metric and a 2-form, an antisymmetric field $B_{\mu\nu}$ (not the same as $g_{[\mu\nu]}$) and plays an important role in string theory compactifications with flux. Recently Hitchin's geometry has been generalized to manifolds with a metric and p-forms by [28] as the appropriate geometry for M theory. Generalized complex geometry has also been instrumental in the geometric Langlands program in physics advanced by [29]. In the next section we will have another interpretation of this doubling of the number of degrees of freedom by working in a suitable phase space involving coordinates and momenta.

2. The large N limit of the $E_7 \times SU(N)$ matrix model and 28-dim bosonic F theory

To construct the $E_7 \times SU(N)$ matrix model, we must first begin with the results of Gunaydin et al. [11] who have shown that there are *no* quadratic $E_{7(7)}$ invariants in the **56**-dim representation of E_7, but instead a real *quartic* invariant I_4 can be built by means of the Freudenthal *ternary* product [12,17] among the elements $X, Y, Z \ldots$ of a Freudenthal algebra $\mathrm{Fr}[O] = J_3[O] \oplus J_3[O] \oplus R^2$ of 56 real dimensions $27 + 27 + 2 = 56$ which is compatible with the 3-grading decomposition of the **56**-dim representation of $E_{7(7)}$ under the $E_{6(6)} \times$ Dilations: $\mathbf{1} \oplus \mathbf{54} \oplus \mathbf{1}$. Hence, the ternary product $X \times Y \times Z \to W$ and a skew-symmetric bilinear form $\langle X, Y\rangle$ yield a quartic E_7 invariant of the form [11]

$$I_4 = \frac{1}{48}\langle (X, X, X), X\rangle = X^{ij} X_{jk} X^{kl} X_{li} - \frac{1}{4} X^{ij} X_{ij} X^{kl} X_{kl} + \frac{1}{96}\epsilon^{ijklmnpq} X_{ij} X_{kl} X_{mn} X_{pq} + \frac{1}{96}\epsilon_{ijklmnpq} X^{ij} X^{kl} X^{mn} X^{pq}. \tag{2.1}$$

where the symplectic invariant of two **56**-dim representations, like the area element in phase space $\int \mathrm{d}p \wedge \mathrm{d}q$, is given by

$$\langle X, Y\rangle = X^{ij} Y_{ij} - X_{ij} Y^{ij} \tag{2.2}$$

and the fundamental **56**-dimensional representation of $E_{7(7)}$ is spanned by the antisymmetric real tensors (bivectors) X^{ij}, X_{ij} built from the $SL(8, R)$ group indices $1 \le i, j \le 8$ and such that the net number of degrees of freedom is

$56 = 28 + 28$ because an $SL(8, R)$ bivector has 28 independent components. There are 28 coordinates X^{ij} and 28 momenta coordinates $X_{ij} = P_{ij}$.

The next step is to construct $E_{7(7)} \times SU(N)$ invariants in the large N limit. This is straightforward once we follow the same steps as in the previous section and after defining the matrix-valued coordinates $\mathcal{M}^A T_A = X^{ijA} T_A$ which take values in the Lie algebra $e_{7(7)} \times su(N)$. The quartic $E_7 \times SU(N)$ invariant which we propose is defined by

$$I_4 = \frac{1}{2} f^E_{AB} f_{CDE} \left[X^{ijA} X^B_{jk} X^{klC} X^D_{li} - \frac{1}{4} X^{ijA} X^B_{ij} X^{klC} X^D_{kl} + \frac{1}{96} \epsilon^{ijklmnpq} X^A_{ij} X^B_{kl} X^C_{mn} X^D_{pq} + \frac{1}{96} \epsilon_{ijklmnpq} X^{ijA} X^{klB} X^{mnC} X^{pqD} \right] \tag{2.3a}$$

where the four-index $SU(N)$ invariant tensor is defined in terms of the structure constants $[T_A, T_B] = f^C_{AB} T_C$ as $\rho_{ABCD} = f^E_{AB} f_{CDE}$. It is not difficult to verify that one can rewrite the terms of Eq. (2.3a) in terms of the trace of $SU(N)$ commutators, since $\mathrm{Trace}(T^A T^B) = \frac{1}{2} \delta^{AB}$, as

$$\mathrm{Trace}([X^{ijA} T_A, X^B_{jk} T_B][X^{klC} T_C, X^D_{li} T_D]) = \frac{1}{2} f^E_{AB} f_{CDE} X^{ijA} X^B_{jk} X^{klC} X^D_{li} \tag{2.3b}$$

and similar equalities hold among the other terms of Eq. (2.3a).

In the large N limit of Eqs. (2.3a) and (2.3b), as explained in [4,5] and summarized in Eqs. (1.1) and (1.8), one has the correspondence $X^{ijA} T_A \to X^{ij}(\sigma^a)$: the $SU(N)$ commutators $[] \to \{\}_{PB}$ and the trace operation $\to \int \mathrm{d}^n \sigma$. Therefore, the large N limit of the expression I_4 in Eq. (2.3a), after rewriting each single term in terms of the trace of $SU(N)$ commutators as indicated by Eq. (2.3b), is given by a generalized nonlinear sigma model action associated with the maps from a four-dim base manifold onto the bivector-valued target space coordinates $X^{ij}(\sigma^a) = -X^{ji}(\sigma^a)$ and momenta $P_{ij}(\sigma^a) = X_{ij}(\sigma^a) = -X_{ji}(\sigma^a)$, that parametrize a 28-complex-dimensional space C^{28} associated with a 56-real-dimensional phase space realization of the Freudenthal algebra $\mathrm{Fr}[O]$ [11,12]. To sum up, the large N limit of the quartic invariant (2.3) furnishes the generalized nonlinear sigma model action on a 56-real-dimensional phase space

$$S = \int [\mathrm{d}^4\sigma] \left[\{X^{ij}, P_{jk}\}\{X^{kl}, P_{li}\} - \frac{1}{4}\{X^{ij}, P_{ij}\}\{X^{kl}, P_{kl}\} + \frac{1}{96} \epsilon^{ijklmnpq} \{P_{ij}, P_{kl}\}\{P_{mn}, P_{pq}\} + \frac{1}{96} \epsilon_{ijklmnpq} \{X^{ij}, X^{kl}\}\{X^{mn}, X^{pq}\} \right] \tag{2.4}$$

and the Poisson brackets are defined with respect to the four coordinates $\sigma^a = \sigma^1, \sigma^2, \sigma^3, \sigma^4$ associated with the four-dim base manifold by

$$\{X^{ij}, X_{jk}\} = \Omega^{ab} \frac{\partial X^{ij}}{\partial \sigma^a} \frac{\partial X_{jk}}{\partial \sigma^b}. \tag{2.5}$$

$\Omega^{a,b}$ is the Poisson symplectic 2-form.

A further analysis of Eq. (2.4) reveals that the last two terms of Eq. (2.4) are zero due to the antisymmetry $\{P_{kl}, P_{ij}\} = -\{P_{ij}, P_{kl}\}$ and the condition $\epsilon^{klijmnpq} = \epsilon^{ijklmnpq}$ (an even permutation of indices). Thus, we are left only with

$$S = \int [\mathrm{d}^4\sigma] \left[\{X^{ij}, P_{jk}\}\{X^{kl}, P_{li}\} - \frac{1}{4}\{X^{ij}, P_{ij}\}\{X^{kl}, P_{kl}\} \right]. \tag{2.6}$$

It remains to be studied whether or not the four-dim base manifold in (2.6) can be identified with the four-dim world-volume of a 3-brane. The action of (2.6) bears a resemblance to the action of (1.8) corresponding to the $3+1$-dimensional world-volume of a 3-brane. Namely, if one can interpret the elements $X, Y, Z \ldots$ of the Freudenthal $\mathrm{Fr}[O]$ algebra in terms of the 28 *complex* coordinates corresponding to the embeddings of a *complexified* $4d$ world-volume (associated with a 3-brane) onto a complexified 12-dim spacetime, and which result from the foliations of the 28-complex-dimensional spacetime into 16-complex-dimensional leaves (like the projective plane $(C \times O)P^2$) along the 12-complex-dimensional spacetime $\mathcal{M}^{12}$. Another picture is naturally provided by studying the compactifications of a

28-complex-dimensional spacetime on 16-complex-dimensional internal spaces, like the projective plane $(C \times O)P^2$ whose isometry group is E_6, yielding Einstein–Yang–Mills actions in 12 complex dimensions. We conjecture that this novel $E_7 \times SU(N)$ matrix model would be the appropriate arena for a bosonic formulation of F theory [9], in the same vein as the formulation of the heterotic string being based upon compactifications of the 26-dim bosonic string on 16-dim lattices.

Concluding, the generalized nonlinear sigma model action (2.6) should describe the global dynamics of a *complexified* 3-brane embedded in 28-complex-dimensional (56-real-dimensional phase space) corresponding to a complexified bosonic formulation of F theory. Identical results can be attained when the phase space coordinates X^{ij}, P_{ij} belong to the complexification of the Freudenthal algebra Fr$[C \times O]$ algebra of $4 \times 28 = 112$ real dimensions. In this case one would have the quaternionic version of the bosonic formulation of F theory in 28-quaternionic-dimensional spaces. The connections between F theory and Jordan algebras of degree 4, $J_4[H]$, have been described by Smith [15].

Finally, we shall present a modification of the Dirac–Nambu–Goto membrane action in terms of a $3 \times 3 \times 3$ cubic matrix H_{abc}. A generalization of a determinant for matrix elements of non-associative Jordan algebra has been provided by Freudenthal, $\det X = \frac{1}{3}(X, X, X)$, in terms of the cubic form. Despite the non-associativity of octonions precluding the ordinary definition of a determinant, another interesting possibility to explore is writing the *cubic* matrix X^{ABC} of $3 \times 3 \times 3 = 27$ entries that matches precisely the number, 27, of independent components of the Jordan 3×3 Hermitian matrices belonging to $J_3[O]$ algebra, and whose hyper-determinant is

$$\text{Det } \mathbf{X} \sim \epsilon^{A_1 A_2 A_3} \epsilon^{B_1 B_2 B_3} \epsilon^{C_1 C_2 C_3} X_{A_1 B_1 C_1} X_{A_2 B_2 C_2} X_{A_3 B_3 C_3}. \tag{2.7}$$

One could then construct a generalization of the Dirac–Nambu–Goto membrane action:

$$S = \int \mathrm{d}^3\sigma [|Det\mathbf{H}|]^{1/3} \tag{2.8}$$

where the hyper-metric $\mathbf{H}$ represented by the $3 \times 3 \times 3$ hyper-matrix (cubic matrix) H_{abc} is defined as the pullback of $H_{\mu_1 \mu_2 \mu_3}$:

$$H_{abc} = H_{\mu_1 \mu_2 \mu_3} \partial_a X^{\mu_1} \partial_b X^{\mu_1} \partial_c X^{\mu_1} \tag{2.9}$$

and the Finslerian-like spacetime interval is of the form

$$(\mathrm{d}s)^2 = [H_{\mu_1 \mu_2 \mu_3} \mathrm{d}x^{\mu_1} \mathrm{d}x^{\mu_2} \mathrm{d}x^{\mu_3}]^{2/3}. \tag{2.10}$$

Finslerian-like geometries [19] are related to W_N geometries. The exceptional (magical) Jordan algebras $J_3[R, C, H, O]$ were instrumental in deciphering important algebraic structures in W_3, W_N gravity [18]. For these reasons, the interplay among W_N algebras, Jordan algebras, Finsler geometry and modified Dirac–Nambu–Goto membrane actions warrants further investigation. For recent studies of M, F theory, see [20].

To conclude, it is worth mentioning some developments related to Jordan exceptional algebras and octonions. The E_7 Cartan quartic invariant (2.1) was used in [30] to construct the entanglement measure associated with the tripartite entanglement of seven quantum bits represented by the group $SL(2, C)^3$ and realized in terms of $2 \times 2 \times 2$ cubic matrices. It was shown by [31] that this tripartite entanglement of seven quantum bits is entirely decoded into the discrete geometry of the octonion Cayley–Fano plane. The analogy between quantum information theory and supersymmetric black holes in $4d$ string theory compactifications was extended further by [31]. The role of Jordan algebras associated with the homogeneous symmetric spaces present in the study of extended supergravities, BPS black holes, quantum attractor flows and automorphic forms can be found in [32]. A nonassociative formulation of bosonic strings for $D = 26$ using Jordan algebras was presented a while back by [33]. The results of this work are the first steps in the construction of nonassociative membranes and 3-brane generalizations based on exceptional Jordan algebras.

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